

FERMIONIC QUANTUM OPERATIONS: A COMPUTATIONAL FRAMEWORK II. EXAMPLES, MONOAXIALITY, SCALINGS

GYULA LAKOS

ABSTRACT. The objective of this series of papers is to recover information regarding the behaviour of FQ operations in the case $n = 2$, and FQ conform-operations in the case $n = 3$. In this second part we show some arithmetically constructible examples of FQ operations ($n = 2$), concentrating on monoaxiality, related extensions, and (hyper)scaling.

INTRODUCTION

Advancing analytical tools for noncommuting sets of operators has a long history. In contrast to the relatively simple case of single matrices cf. [2], [10], where the spectrum plays a decisive role, the study of noncommuting sets of operators is much more difficult. Systematic approaches bring forth various organization principles, like the use of the joint spectrum [11], powerful applications of perturbation expansions [5] (also see references therein), operator orderings [8] (with much prehistory in quantum physics), application of Clifford algebras [1] along with efforts to integrate various approaches, cf. [4]. The study of concrete operator functions is also developing, especially operator means, see e. g. [3], and in preserver problems, which is a sort of an invariant theory for noncommutative systems, cf. [9]. Combining Clifford systems and free analysis, we intend to further the idea of using Clifford systems as base systems, whose perturbations can be investigated: More specifically, FQ operations were introduced in [6] as some kind of quasi linear algebraic tools for non-commuting operators. These can be approached on global (analytic) and local / punctual (formal) level. The objective of this series of papers (cf. [7]) is to recover information regarding the behaviour of FQ operations in the case $n = 2$, and FQ conform-operations in the case $n = 3$. In this second part we show some arithmetically constructible examples of FQ operations ($n = 2$), concentrating on monoaxiality, related extensions, and (hyper)scaling. Here, in contrast to [7], we consider only natural (conjugation-invariant) FQ operations with sign-linearity. Moreover, we aim here to orthogonal invariant FQ operations. The outline of the paper is as follows. Section 1 contains some arithmetical constructions of FQ operations. In particular, we construct an FQ orthogonalization procedure. Due to this arithmeticity (as opposed to constructions by iterative methods) examples here will be rather simple; and some of them are arguably the simplest ones. The emphasis is on showing up some “major” types. There are many similar ones with various tradeoffs in their behaviour. Some of them have less than perfect properties (like no involutivity or idempotence) but they might serve as initial objects in other constructions. In Section 2 we consider the axial extension procedure. In Section 3 we consider hyperscaling conditions and see why they are too strong.

2010 *Mathematics Subject Classification.* Primary: 47L99. Secondary: 15B99, 81Q99.

Key words and phrases. Clifford systems, formal power series, calculus of non-commuting operators.

1. SOME ARITHMETICALLY CONSTRUCTIBLE EXAMPLES

In this section we will consider certain analytic FQ operations. Our computations will be valid when the involved terms (\mathcal{A}_C , $\mathcal{B}^{-1/2}$, etc.) are well-defined, but always valid on the formal domain. In this section all FQ operations will be symmetric, and, in fact, orthogonal invariant. Whenever we consider an operation we will give its first-order expansion terms relative to the mixed base of [7], which tells much about the character of the operation. Beyond that, in order to distinguish the various cases, we will not give terms of higher order, which are hard to interpret, but we indicate their scalar scaling properties (i. e. possible scaling variables in the mixed base). Fortunately, scalar scaling properties are rather easy to check or disprove in concrete cases.

1.1. In if $A, B \in \mathfrak{A}$, and the straight segment $\{(1-t)A + tB : t \in [0, 1]\}$ lies in \mathfrak{A}^\star , then one defines the geometric mean

$$(A \star B)^{1/2} := \int_{t=0}^{2\pi} \frac{1}{A^{-1} \cos^2 t + B^{-1} \sin^2 t} \frac{dt}{2\pi},$$

which is also the common value

$$A(A^{-1}B)^{1/2} = B(B^{-1}A)^{1/2} = (AB^{-1})^{1/2}B = (BA^{-1})^{1/2}A.$$

One can see that if A and B are skew-involutions, then $(A \star B)^{1/2}$ is also a skew-involution, furthermore,

$$(A \star B)^{1/2} = \text{pol } \frac{1}{2}(A + B);$$

where we have used the notation $\text{pol } X = X(-X^2)^{-1/2}$. Moreover, in that case, A and B are conjugates of each other by $(A \star B)^{1/2}$. We will occasionally use the notation $|A| = (-A^2)^{1/2}$.

1.2. We define the pseudoscalar FQ operations left axis \mathcal{A}_L by

$$\mathcal{A}_L(A_1, A_2) := \text{pol } -A_1 A_2^{-1} = \text{pol } A_2 A_1^{-1};$$

right axis \mathcal{A}_R

$$\mathcal{A}_R(A_1, A_2) := \text{pol } A_2^{-1} A_1 = \text{pol } -A_1^{-1} A_2;$$

and (central) axis \mathcal{A}_C by

$$\mathcal{A}_C(A_1, A_2) := (\mathcal{A}_L(A_1, A_2) \star \mathcal{A}_R(A_1, A_2))^{1/2}.$$

Furthermore we define the scalar FQ operation biaxiality \mathcal{B} as

$$\mathcal{B}(A_1, A_2) := (-\mathcal{A}_L(A_1, A_2) \cdot \mathcal{A}_R(A_1, A_2))^{1/2};$$

in this case

$$\mathcal{B}(A_1, A_2)^{-1} = (-\mathcal{A}_R(A_1, A_2) \cdot \mathcal{A}_L(A_1, A_2))^{1/2}.$$

One can see that the corresponding terms in their (formal) expansion are

$$\begin{aligned} \mathcal{A}_L : \quad & \hat{\mathbf{P}}_0^{[12]} = [1] & \hat{\mathbf{P}}_1^{[12]} &= [0 \ 2 \ 0 \ 0 \ 0 \ 0 \ 2 \ 0]; \\ \mathcal{A}_R : \quad & \hat{\mathbf{P}}_0^{[12]} = [1] & \hat{\mathbf{P}}_1^{[12]} &= [0 \ -2 \ 0 \ 0 \ 0 \ 0 \ 2 \ 0]; \\ \mathcal{A}_C : \quad & \hat{\mathbf{P}}_0^{[12]} = [1] & \hat{\mathbf{P}}_1^{[12]} &= [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 2 \ 0]; \end{aligned}$$

and

$$\begin{aligned} \mathcal{B} : \quad & \hat{\mathbf{P}}_0^{[0]} = [1] & \hat{\mathbf{P}}_1^{[0]} &= [0 \ 2 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]; \\ \mathcal{B}^{-1} : \quad & \hat{\mathbf{P}}_0^{[0]} = [1] & \hat{\mathbf{P}}_1^{[0]} &= [0 \ -2 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]. \end{aligned}$$

It is immediate but useful to keep in mind that, using the notation $A_1 = \text{Id}_1$, $A_2 = \text{Id}_2$,

$$(1) \ \mathcal{A}_L \mathcal{A}_1 = -A_1 \mathcal{A}_R, \quad \mathcal{A}_L \mathcal{A}_2 = -A_2 \mathcal{A}_R;$$

- (2) $\mathcal{A}_L = \mathcal{B}^{\pm 1} \mathcal{A}_L \mathcal{B}^{\pm 1} = \mathcal{B} \mathcal{A}_C = \mathcal{A}_C \mathcal{B}^{-1} = \mathcal{B}^2 \mathcal{A}_R = \mathcal{B} \mathcal{A}_R \mathcal{B}^{-1} = \mathcal{A}_R \mathcal{B}^{-2} = -\mathcal{A}_C \mathcal{A}_R \mathcal{A}_C$;
- (3) $\mathcal{A}_R = \mathcal{B}^{\pm 1} \mathcal{A}_R \mathcal{B}^{\pm 1} = \mathcal{B}^{-1} \mathcal{A}_C = \mathcal{A}_C \mathcal{B} = \mathcal{B}^{-2} \mathcal{A}_L = \mathcal{B}^{-1} \mathcal{A}_L \mathcal{B} = \mathcal{A}_L \mathcal{B}^2 = -\mathcal{A}_C \mathcal{A}_L \mathcal{A}_C$;
- (4) $\mathcal{A}_C = \mathcal{B}^{\pm 1} \mathcal{A}_C \mathcal{B}^{\pm 1} = \mathcal{B} \mathcal{A}_R = \mathcal{A}_R \mathcal{B}^{-1} = \mathcal{B}^{-1} \mathcal{A}_L = \mathcal{A}_L \mathcal{B}$;
- (5) $\mathcal{B} = -\mathcal{A}_L \mathcal{B}^{-1} \mathcal{A}_L = -\mathcal{A}_C \mathcal{B}^{-1} \mathcal{A}_C = -\mathcal{A}_R \mathcal{B}^{-1} \mathcal{A}_R = -\mathcal{A}_C \mathcal{A}_R = -\mathcal{A}_L \mathcal{A}_C$;
- (6) $\mathcal{B}^{-1} = -\mathcal{A}_L \mathcal{B} \mathcal{A}_L = -\mathcal{A}_C \mathcal{B} \mathcal{A}_C = -\mathcal{A}_R \mathcal{B} \mathcal{A}_R = -\mathcal{A}_R \mathcal{A}_C = -\mathcal{A}_C \mathcal{A}_L$.

One can also see that $\mathcal{A}_L, \mathcal{A}_C, \mathcal{A}_R, \mathcal{B}, \mathcal{B}^{-1}$ satisfy scaling invariances in $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5$.

1.3. Taking multiplication positionwise, $\mathcal{A}_L(A_1, A_2) \cdot (A_1, A_2) = -(A_1, A_2) \cdot \mathcal{A}_R(A_1, A_2)$. This allows us to define the axial turn operation \mathcal{T} as

$$\mathcal{T}(A_1, A_2) := \mathcal{A}_L(A_1, A_2) \cdot (-A_2, A_1) = (A_2, -A_1) \cdot \mathcal{A}_R(A_1, A_2).$$

The corresponding terms in its expansion are

$$\mathcal{T} : \quad \hat{\mathbf{P}}_0^{[1]} = [1] \quad \hat{\mathbf{P}}_1^{[1]} = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

One can see

$$\mathcal{O}^{\text{fSy}}(A_1, A_2) = \frac{1}{2}((A_1, A_2) + \mathcal{T}(A_1, A_2)),$$

where \mathcal{O}^{fSy} is the canonical realization of the symmetric conform-orthogonalization procedure from [6]. The corresponding terms in its expansion are

$$\mathcal{O}^{\text{fSy}} : \quad \hat{\mathbf{P}}_0^{[1]} = [1] \quad \hat{\mathbf{P}}_1^{[1]} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

We define the anticonform orthogonalization operation $\mathcal{O}^{\text{afSy}}$ by

$$\mathcal{O}^{\text{afSy}}(A_1, A_2) := (-\mathcal{O}^{\text{fSy}}(A_1, A_2)_1^{-1}, -\mathcal{O}^{\text{fSy}}(A_1, A_2)_2^{-1}).$$

$\mathcal{O}^{\text{afSy}}$ also produces a floating Clifford system, except this operation is not bivariant but antivariant. ($\mathcal{O}^{\text{afSy}}$ is set up so that it is Clifford conservative.) $\mathcal{O}^{\text{afSy}}$ can also be realized by a closed integral formula, cf. [6]. The corresponding terms in its expansion are

$$\mathcal{O}^{\text{afSy}} : \quad \hat{\mathbf{P}}_0^{[1]} = [1] \quad \hat{\mathbf{P}}_1^{[1]} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

One can check that the following hold:

- (1) $\mathcal{A}_L \circ \mathcal{T} = \mathcal{A}_L, \quad \mathcal{A}_C \circ \mathcal{T} = \mathcal{A}_C, \quad \mathcal{A}_R \circ \mathcal{T} = \mathcal{A}_R, \quad \mathcal{B} \circ \mathcal{T} = \mathcal{B}$;
- (2) $\mathcal{A}_L \circ \mathcal{O}^{\text{fSy}} = \mathcal{A}_L, \quad \mathcal{A}_C \circ \mathcal{O}^{\text{fSy}} = \mathcal{A}_C, \quad \mathcal{A}_R \circ \mathcal{O}^{\text{fSy}} = \mathcal{A}_R, \quad \mathcal{B} \circ \mathcal{O}^{\text{fSy}} = \mathcal{B}$;
- (3) $\mathcal{A}_L \circ \mathcal{O}^{\text{afSy}} = \mathcal{A}_R, \quad \mathcal{A}_C \circ \mathcal{O}^{\text{afSy}} = \mathcal{A}_C, \quad \mathcal{A}_R \circ \mathcal{O}^{\text{afSy}} = \mathcal{A}_L, \quad \mathcal{B} \circ \mathcal{O}^{\text{afSy}} = \mathcal{B}^{-1}$.

One can also see that \mathcal{T} satisfies scaling invariances in variables $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5$; \mathcal{O}^{fSy} satisfies scaling invariances in variables $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_5$; $\mathcal{O}^{\text{afSy}}$ satisfies scaling invariances in variables $\hat{r}_1, \hat{r}_2, \hat{r}_3, \tilde{r}_4$. (Here \tilde{r}_4 and \tilde{r}_5 refer to some mixed scaling conditions in \hat{r}_4/\hat{r}_5 .)

1.4. We say that the pair (A_1, A_2) is monoaxial if $\mathcal{A}_L(A_1, A_2)$ and $\mathcal{A}_R(A_1, A_2)$, and hence, $\mathcal{A}_C(A_1, A_2)$, are equal to each other. This happens if and only if $\mathcal{B}(A_1, A_2) = 1$. Another equivalent formulation is that $\mathcal{A}_C(A_1, A_2)$ anticommutes with A_1, A_2 . Indeed, the identities $A_1 \mathcal{A}_R(A_1, A_2) A_1^{-1} = \mathcal{A}_L(A_1, A_2)$ and $A_2 \mathcal{A}_R(A_1, A_2) A_2^{-1} = \mathcal{A}_L(A_1, A_2)$ imply that in the monoaxial case A_1 and A_2 commute with the axes, i. e. with the axis. Conversely, if $\mathcal{A}_C(A_1, A_2)$ commutes with A_1, A_2 then it commutes with $\mathcal{A}_L(A_1, A_2) = \text{pol} - A_1 A_2^{-1}$; on the other hand, as a general rule, we know that $\mathcal{A}_L(A_1, A_2)$ conjugated by $\mathcal{A}_C(A_1, A_2)$ is $\mathcal{A}_R(A_1, A_2)$; consequently $\mathcal{A}_L(A_1, A_2) = \mathcal{A}_R(A_1, A_2)$.

1.5. **Theorem.** *If (A_1, A_2) is monoaxial with axis $\mathcal{A}_C = \mathcal{A}_C(A_1, A_2)$, and*

$$1 + \tilde{r}_3^x := \left(-\frac{A_1^2 + A_2^2 - A_1 \mathcal{A}_C A_2 + A_2 \mathcal{A}_C A_1}{4} \right)^{1/2}$$

exists; then, we claim,

$$Q_1 = \text{pol } \mathcal{O}^{\text{fSy}}(A_1, A_2)_1, \quad Q_2 = \text{pol } \mathcal{O}^{\text{fSy}}(A_1, A_2)_2$$

forms a Clifford system; and with $\tilde{r}_4^x := -A_1Q_1 + Q_2A_2$, the decomposition

$$(A_1, A_2) = ((1 + \tilde{r}_3^x + \tilde{r}_4^x)Q_1, (1 + \tilde{r}_3^x - \tilde{r}_4^x)Q_2)$$

is valid. (Actually, this is the circular decomposition with respect to (Q_1, Q_2) .) Here Q_1, Q_2 commute with \tilde{r}_3^x and anticommute with \mathcal{A}_C and \tilde{r}_4^x ; moreover, $\mathcal{A}_C = Q_1Q_2$.

Furthermore,

$$\mathcal{O}^{\text{fSy}}(A_1, A_2) = ((1 + \tilde{r}_3^x)Q_1, (1 + \tilde{r}_3^x)Q_2),$$

and

$$\mathcal{O}_m^{\text{Sy}}(A_1, A_2) = (Q_1, Q_2);$$

where the latter equation is understood such that it provides an analytical realization of $\underline{\mathcal{O}}^{\text{Sy}}$ (for some monoaxial elements, though).

Proof. From monoaxiality one can deduce

$$(\mathcal{O}^{\text{fSy}}(A_1, A_2)_1)^2 = (\mathcal{O}^{\text{fSy}}(A_1, A_2)_2)^2 = \frac{1}{4} (A_1^2 + A_2^2 - A_1\mathcal{A}_CA_2 + A_2\mathcal{A}_CA_1).$$

After that, it is a straightforward computation. \square

1.6. We say that a vectorial FQ operation is monoaxial if its result is always monoaxial (e. g.: FQ orthogonalizations). We define the vectorial FQ operations left monoaxialization

$$\mathcal{M}_L(A_1, A_2) := (A_1, A_2) \cdot \mathcal{B}(A_1, A_2)^{-1};$$

right monoaxialization,

$$\mathcal{M}_R(A_1, A_2) := \mathcal{B}(A_1, A_2)^{-1} \cdot (A_1, A_2);$$

and central monoaxialization

$$\mathcal{M}_C(A_1, A_2) := \mathcal{B}(A_1, A_2)^{-1/2} \cdot (A_1, A_2) \cdot \mathcal{B}(A_1, A_2)^{-1/2}.$$

These are conjugates of each other. It is easy to see that $\mathcal{M}_L, \mathcal{M}_R, \mathcal{M}_C$ are monoaxial with axes $\mathcal{A}_L, \mathcal{A}_R, \mathcal{A}_C$, respectively; and they act trivially on monoaxial pairs.

The corresponding terms in their (formal) expansion are

$$\begin{aligned} \mathcal{M}_L : \quad \hat{\mathbf{P}}_0^{[1]} &= [1] & \hat{\mathbf{P}}_1^{[1]} &= \begin{bmatrix} -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}; \\ \mathcal{M}_R : \quad \hat{\mathbf{P}}_0^{[1]} &= [1] & \hat{\mathbf{P}}_1^{[1]} &= \begin{bmatrix} 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}; \\ \mathcal{M}_C : \quad \hat{\mathbf{P}}_0^{[1]} &= [1] & \hat{\mathbf{P}}_1^{[1]} &= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

We define the axial conjugation operation \mathcal{C} as

$$\mathcal{C}(A_1, A_2) := \mathcal{A}_C(A_1, A_2) \cdot (A_1, A_2) \cdot \mathcal{A}_C(A_1, A_2) = \mathcal{B}(A_1, A_2)^{-1} \cdot (A_1, A_2) \cdot \mathcal{B}(A_1, A_2)^{-1}.$$

In terms of its expansion

$$\mathcal{C} : \quad \hat{\mathbf{P}}_0^{[1]} = [1] \quad \hat{\mathbf{P}}_1^{[1]} = \begin{bmatrix} -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

One can easily check that with the choices $X, Y = L, C, R$ and conventions $-L = R, -C = C, -R = L$, the following hold:

- (1) $\mathcal{A}_Y \circ \mathcal{M}_X = \mathcal{A}_X, \quad \mathcal{A}_{-X} \circ \mathcal{C} = \mathcal{A}_X;$
- (2) $\mathcal{B} \circ \mathcal{M}_X = 1, \quad \mathcal{B} \circ \mathcal{C} = \mathcal{B}^{-1};$
- (3) $\mathcal{T} \circ \mathcal{M}_X = \mathcal{M}_X \circ \mathcal{T}, \quad \mathcal{T} \circ \mathcal{C} = \mathcal{C} \circ \mathcal{T};$
- (4) $\mathcal{O}^{\text{fSy}} \circ \mathcal{M}_X = \mathcal{M}_X \circ \mathcal{O}^{\text{fSy}}, \quad \mathcal{O}^{\text{fSy}} \circ \mathcal{C} = \mathcal{C} \circ \mathcal{O}^{\text{fSy}};$
- (5) $\mathcal{O}^{\text{afSy}} \circ \mathcal{M}_X = \mathcal{M}_{-X} \circ \mathcal{O}^{\text{afSy}}, \quad \mathcal{O}^{\text{afSy}} \circ \mathcal{C} = \mathcal{C} \circ \mathcal{O}^{\text{afSy}};$

- (6) $\mathcal{M}_Y \circ \mathcal{M}_X = \mathcal{M}_{-X} \circ \mathcal{C} = \mathcal{C} \circ \mathcal{M}_X = \mathcal{M}_X$, $\mathcal{C} \circ \mathcal{C} = \text{Id}$;
 (7) $\mathcal{B}(A_1, A_2) = 1 \Leftrightarrow \mathcal{M}_X(A_1, A_2) = (A_1, A_2)$.

Furthermore, $\mathcal{M}_L, \mathcal{M}_C, \mathcal{M}_R, \mathcal{C}$ satisfy scaling invariances in $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5$.

1.7. At this point one can easily define an orthogonalization procedure by

$$\mathcal{O}^{\text{mSy}} := \mathcal{O}_m^{\text{Sy}} \circ \mathcal{M}_C.$$

Indeed, after monoaxialization, $\mathcal{O}_m^{\text{Sy}}$ can be applied. In its expansion, the corresponding terms of first-order are

$$\mathcal{O}^{\text{mSy}} : \quad \hat{\mathbf{P}}_0^{[1]} = [1] \quad \hat{\mathbf{P}}_1^{[1]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

From the definition it is easy to see that

- (1) $\mathcal{A}_Y \circ \mathcal{O}^{\text{mSy}} = \mathcal{A}_C$, $\mathcal{B} \circ \mathcal{O}^{\text{mSy}} = 1$;
 (2) $\mathcal{O}^{\text{mSy}} = \mathcal{O}^{\text{mSy}} \circ \mathcal{O}^{\text{fSy}} = \mathcal{O}^{\text{mSy}} \circ \mathcal{O}^{\text{afSy}} = \mathcal{O}^{\text{mSy}} \circ \mathcal{T} = \mathcal{O}^{\text{mSy}} \circ \mathcal{M}_C = \mathcal{O}^{\text{mSy}} \circ \mathcal{C}$.

Now, \mathcal{O}^{mSy} has the same expansion in first order as $\underline{\mathcal{Q}}^{\text{Sy}}$, but they already differ in second order. More informatively, one can see that (relative to the mixed base) $\underline{\mathcal{Q}}^{\text{Sy}}$ has a single scaling invariance, in the variable \hat{r}_3 , i. e. scalar homogeneity. On the other hand, one can show that \mathcal{O}^{mSy} has scaling invariances in the variables $\hat{r}_1, \hat{r}_2, \hat{r}_3$ (but it does not satisfy metric trace commutativity then, of course).

1.8. In order to have a more explicit form, we define the scalar FQ operation axial length (squared) \mathcal{L}^{fx} by

$$\mathcal{L}^{\text{fx}}(A_1, A_2) := \frac{1}{4} (-A_1 \mathcal{B}^{-1} A_1 - A_2 \mathcal{B}^{-1} A_2 + A_1 \mathcal{A}_C A_2 - A_2 \mathcal{A}_C A_1),$$

axial volume \mathcal{V}^{fm} by

$$\mathcal{V}^{\text{fm}}(A_1, A_2) := \mathcal{B}^{-1/2} \mathcal{L}^{\text{fx}}(A_1, A_2) \mathcal{B}^{-1/2},$$

and the axial pseudodeterminant \mathcal{D}^{fx} as

$$\mathcal{D}^{\text{fx}}(A_1, A_2) := \frac{1}{4} (A_1 \mathcal{A}_C A_1 + A_2 \mathcal{A}_C A_2 + A_1 \mathcal{B}^{-1} A_2 - A_2 \mathcal{B}^{-1} A_1).$$

where $\mathcal{B} = \mathcal{B}(A_1, A_2)$, and $\mathcal{A}_C = \mathcal{A}_C(A_1, A_2)$. Terms in their expansion are

$$\begin{aligned} \mathcal{L}^{\text{fx}} : \quad \hat{\mathbf{P}}_0^{[0]} &= [1] \quad \hat{\mathbf{P}}_1^{[0]} = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \\ \mathcal{V}^{\text{fm}} : \quad \hat{\mathbf{P}}_0^{[0]} &= [1] \quad \hat{\mathbf{P}}_1^{[0]} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \\ \mathcal{D}^{\text{fx}} : \quad \hat{\mathbf{P}}_0^{[12]} &= [1] \quad \hat{\mathbf{P}}_1^{[12]} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}. \end{aligned}$$

One can easily check that the following hold:

- (1) $\mathcal{D}^{\text{fx}} = \mathcal{A}_L \mathcal{L}^{\text{fx}} = \mathcal{L}^{\text{fx}} \mathcal{A}_R$, $\mathcal{L}^{\text{fx}} \mathcal{B}^{-1} = -\mathcal{D}^{\text{fx}} \mathcal{A}_C$, $\mathcal{B}^{-1} \mathcal{L}^{\text{fx}} = -\mathcal{A}_C \mathcal{D}^{\text{fx}}$;
 (2) $\mathcal{L}^{\text{fx}} \circ \mathcal{O}^{\text{fSy}} = \mathcal{L}^{\text{fx}}$, $\mathcal{L}^{\text{fx}} \circ \mathcal{O}^{\text{mSy}} = 1$, $\mathcal{L}^{\text{fx}} \circ \mathcal{O}^{\text{afSy}} = (\mathcal{L}^{\text{fx}})^{-1}$;
 (3) $\mathcal{V}^{\text{fm}} \circ \mathcal{O}^{\text{fSy}} = \mathcal{V}^{\text{fm}}$, $\mathcal{V}^{\text{fm}} \circ \mathcal{O}^{\text{mSy}} = 1$, $\mathcal{V}^{\text{fm}} \circ \mathcal{O}^{\text{afSy}} = (\mathcal{V}^{\text{fm}})^{-1}$;
 (4) $\mathcal{D}^{\text{fx}} \circ \mathcal{O}^{\text{fSy}} = \mathcal{D}^{\text{fx}}$, $\mathcal{D}^{\text{fx}} \circ \mathcal{O}^{\text{mSy}} = \mathcal{A}_C$, $\mathcal{D}^{\text{fx}} \circ \mathcal{O}^{\text{afSy}} = -(\mathcal{D}^{\text{fx}})^{-1}$;
 (5) $\mathcal{O}^{\text{mSy}} = (\mathcal{V}^{\text{fm}})^{-1/2} \mathcal{B}^{-1/2} \mathcal{O}^{\text{fSy}} \mathcal{B}^{-1/2} = \mathcal{B}^{-1/2} \mathcal{O}^{\text{fSy}} \mathcal{B}^{-1/2} (\mathcal{V}^{\text{fm}})^{-1/2}$
 $= \mathcal{B}^{-1/2} (\mathcal{L}^{\text{fx}} \mathcal{B}^{-1})^{-1/2} \mathcal{O}^{\text{fSy}} \mathcal{B}^{-1/2} = \mathcal{B}^{-1/2} \mathcal{O}^{\text{fSy}} (\mathcal{B}^{-1} \mathcal{L}^{\text{fx}})^{-1/2} \mathcal{B}^{-1/2}.$

One can also prove that $\mathcal{L}^{\text{fx}}, \mathcal{V}^{\text{fm}}, \mathcal{D}^{\text{fx}}$ are scaling invariant in variables $\hat{r}_1, \hat{r}_2, \hat{r}_3$.

Notice that \mathcal{L}^{fx} and \mathcal{D}^{fx} has some variants given by

$$\begin{aligned} \mathcal{L}^{\text{x}}(A_1, A_2) &:= \frac{1}{2} (-A_1 \mathcal{B}^{-1} A_1 - A_2 \mathcal{B}^{-1} A_2), \\ \mathcal{L}^{\text{cx}}(A_1, A_2) &:= \frac{1}{2} (A_1 \mathcal{A}_C A_2 - A_2 \mathcal{A}_C A_1), \end{aligned}$$

$$\mathcal{D}^x(A_1, A_2) := \frac{1}{2} (A_1 \mathcal{B}^{-1} A_2 - A_2 \mathcal{B}^{-1} A_1) = \mathcal{A}_L \cdot \mathcal{L}^{\text{cx}}(A_1, A_2) = \mathcal{L}^{\text{cx}}(A_1, A_2) \cdot \mathcal{A}_R,$$

$$\mathcal{D}^{\text{cx}}(A_1, A_2) := \frac{1}{2} (A_1 \mathcal{A}_C A_1 + A_2 \mathcal{A}_C A_2) = \mathcal{A}_L \cdot \mathcal{L}^x(A_1, A_2) = \mathcal{L}^x(A_1, A_2) \cdot \mathcal{A}_R,$$

where $\mathcal{A}_L = \mathcal{A}_L(A_1, A_2)$, $\mathcal{A}_R = \mathcal{A}_R(A_1, A_2)$; which can be related to the even simpler operations

$$\mathcal{L}(A_1, A_2) := \frac{1}{2} (-A_1^2 - A_2^2),$$

$$\mathcal{D}(A_1, A_2) := \frac{1}{2} (A_1 A_2 - A_2 A_1).$$

Here \mathcal{L}^x and \mathcal{D}^{cx} satisfy scalings in variables $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_5$; \mathcal{L}^{cx} and \mathcal{D}^x satisfy scalings in variables $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4$; \mathcal{L} satisfies scalings in variables $\hat{r}_2, \hat{r}_3, \hat{r}_5$ and an \hat{r}_1/\hat{r}_4 mixed condition; \mathcal{D} satisfies scalings in variables $\hat{r}_2, \hat{r}_3, \hat{r}_4$ and an \hat{r}_1/\hat{r}_5 mixed condition (not detailed here).

One can consider the polarizations $\text{pol } \mathcal{D}^{\text{fx}}, \text{pol } \mathcal{D}^x, \text{pol } \mathcal{D}^{\text{cx}}$ and $\mathcal{A}_D := \text{pol } \mathcal{D}$. They have the same first-order expansions as \mathcal{A}_C . $\text{pol } \mathcal{D}^{\text{fx}}$ scales in $\hat{r}_1, \hat{r}_2, \hat{r}_3$; $\text{pol } \mathcal{D}^x$ scales in $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4$; $\text{pol } \mathcal{D}^{\text{cx}}$ scales in $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_5$; $\mathcal{A}_D = \text{pol } \mathcal{D}$ scales in $\hat{r}_2, \hat{r}_3, \hat{r}_4$. Or, one can take the “volumes” $|\mathcal{D}^{\text{fx}}|, |\mathcal{D}^x|, |\mathcal{D}^{\text{cx}}|$ and $\mathcal{V} := |\mathcal{D}|$. They have the same first-order expansions as \mathcal{V}^{fm} . $|\mathcal{D}^{\text{fx}}|$ scales in $\hat{r}_1, \hat{r}_2, \hat{r}_3$; $\text{pol } \mathcal{D}^x$ scales in $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4$; $|\mathcal{D}^{\text{cx}}|$ scales in $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_5$; $\mathcal{V} = |\mathcal{D}|$ scales in $\hat{r}_2, \hat{r}_3, \hat{r}_4$. (Remark: $\mathcal{V}^{\text{fm}} \neq |\mathcal{D}^{\text{fx}}|$, they differ in third order, or, more qualitatively, \mathcal{V}^{fm} commutes with \mathcal{A}_C multiplicatively, but $|\mathcal{D}^{\text{fx}}|$ does not.)

1.9. We can define the vectorial FQ operation axial unitalization \mathcal{U}^{fx} by

$$\mathcal{U}^{\text{fx}}(A_1, A_2) := \mathcal{B}^{1/2} (\mathcal{V}^{\text{fm}})^{-1/4} \mathcal{B}^{-1/2} \cdot (A_1, A_2) \cdot \mathcal{B}^{-1/2} (\mathcal{V}^{\text{fm}})^{-1/4} \mathcal{B}^{1/2};$$

and the vectorial FQ operation axial amplitude inversion \mathcal{K}^{fx} by

$$\mathcal{K}^{\text{fx}}(A_1, A_2) := \mathcal{B}^{1/2} (\mathcal{V}^{\text{fm}})^{-1/2} \mathcal{B}^{-1/2} \cdot (A_1, A_2) \cdot \mathcal{B}^{-1/2} (\mathcal{V}^{\text{fm}})^{-1/2} \mathcal{B}^{1/2};$$

where $\mathcal{B} = \mathcal{B}(A_1, A_2)$, $\mathcal{V}^{\text{fm}} = \mathcal{V}^{\text{fm}}(A_1, A_2)$. Terms in their expansion are

$$\mathcal{U}^{\text{fx}} : \quad \hat{\mathbf{P}}_0^{[1]} = [1] \quad \hat{\mathbf{P}}_1^{[1]} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix};$$

$$\mathcal{K}^{\text{fx}} : \quad \hat{\mathbf{P}}_0^{[1]} = [1] \quad \hat{\mathbf{P}}_1^{[1]} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

One can see that

- (1) $\mathcal{K}^{\text{fx}} \circ \mathcal{K}^{\text{fx}} = \text{Id}$, $\mathcal{U}^{\text{fx}} \circ \mathcal{U}^{\text{fx}} = \mathcal{U}^{\text{fx}} \circ \mathcal{K}^{\text{fx}} = \mathcal{K}^{\text{fx}} \circ \mathcal{U}^{\text{fx}} = \mathcal{U}^{\text{fx}}$;
- (2) $\mathcal{A}_L \circ \mathcal{U}^{\text{fx}} = \mathcal{A}_L$, $\mathcal{A}_C \circ \mathcal{U}^{\text{fx}} = \mathcal{A}_C$, $\mathcal{A}_R \circ \mathcal{U}^{\text{fx}} = \mathcal{A}_R$, $\mathcal{B} \circ \mathcal{U}^{\text{fx}} = \mathcal{B}$;
- (3) $\mathcal{A}_L \circ \mathcal{K}^{\text{fx}} = \mathcal{A}_L$, $\mathcal{A}_C \circ \mathcal{K}^{\text{fx}} = \mathcal{A}_C$, $\mathcal{A}_R \circ \mathcal{K}^{\text{fx}} = \mathcal{A}_R$, $\mathcal{B} \circ \mathcal{K}^{\text{fx}} = \mathcal{B}$;
- (4) \mathcal{U}^{fx} and \mathcal{K}^{fx} commutes with $\mathcal{M}_L, \mathcal{M}_C, \mathcal{M}_R, \mathcal{C}, \mathcal{O}^{\text{fSy}}, \mathcal{O}^{\text{afSy}}, \mathcal{T}$ with respect to \circ ;
- (5) $\mathcal{O}^{\text{mSy}} = \mathcal{U}^{\text{fx}} \circ \mathcal{O}^{\text{fSy}} \circ \mathcal{M}_C = \mathcal{U}^{\text{fx}} \circ \mathcal{O}^{\text{afSy}} \circ \mathcal{M}_C$, $\mathcal{K}^{\text{fx}} \circ \mathcal{O}^{\text{fSy}} \circ \mathcal{M}_C = \mathcal{O}^{\text{afSy}} \circ \mathcal{M}_C$;
- (6) $\mathcal{L}^{\text{fx}} \circ \mathcal{U}^{\text{fx}} = \mathcal{B}$, $\mathcal{D}^{\text{fx}} \circ \mathcal{U}^{\text{fx}} = \mathcal{A}_C$;
- (7) $\mathcal{V}^{\text{fm}} \circ \mathcal{K}^{\text{fx}} = (\mathcal{V}^{\text{fm}})^{-1}$, $\mathcal{V}^{\text{fm}} \circ \mathcal{U}^{\text{fx}} = 1$; $\mathcal{V}^{\text{fm}}(A_1, A_2) = 1 \Leftrightarrow \mathcal{U}^{\text{fx}}(A_1, A_2) = (A_1, A_2)$.

Furthermore, \mathcal{U}^{fx} and \mathcal{K}^{fx} has scaling invariances in variables $\hat{r}_1, \hat{r}_2, \hat{r}_3$.

1.10. We see that we have three commuting sets of actions $\mathcal{M}_C, \mathcal{C}$ and $\mathcal{U}^{\text{fx}}, \mathcal{K}^{\text{fx}}$ and $\mathcal{O}^{\text{fSy}}, \mathcal{T}$; acting principally in variables \hat{r}_1, \hat{r}_2 and \hat{r}_3 and \hat{r}_4, \hat{r}_5 respectively. We can take the composition $\mathcal{O}^{\text{mSy}} = \mathcal{M}_C \circ \mathcal{U}^{\text{fx}} \circ \mathcal{O}^{\text{fSy}}$ which we have already seen. But one can also define

$$\mathcal{I}_C^{\text{fy}} := \mathcal{C} \circ \mathcal{K}^{\text{fx}} \circ \mathcal{T}.$$

It has expansion terms

$$\mathcal{I}_C^{\text{fy}} : \quad \hat{\mathbf{P}}_0^{[1]} = [1] \quad \hat{\mathbf{P}}_1^{[1]} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

It is an involution,

$$(\mathcal{I}_C^{\text{fy}})^2 = \text{Id}.$$

Here $\mathcal{I}_C^{\text{fy}}$ has scaling invariances in variables $\hat{r}_1, \hat{r}_2, \hat{r}_3$. One can check that

$$\mathcal{I}_C^{\text{fy}}(A_1, A_2) = (\mathcal{B}(A_1, A_2) \star \mathcal{L}^{\text{fx}}(A_1, A_2))^{-1/2} \cdot \mathcal{T}(A_1, A_2) \cdot (\mathcal{B}(A_1, A_2) \star \mathcal{L}^{\text{fx}}(A_1, A_2))^{-1/2}.$$

One can also define

$$\begin{aligned}\mathcal{I}_L^{\text{fy}}(A_1, A_2) &:= \mathcal{B}(A_1, A_2)^{-1} \cdot \mathcal{T}(A_1, A_2) \cdot \mathcal{L}^{\text{fx}}(A_1, A_2)^{-1}, \\ \mathcal{I}_R^{\text{fy}}(A_1, A_2) &:= \mathcal{L}^{\text{fx}}(A_1, A_2)^{-1} \cdot \mathcal{T}(A_1, A_2) \cdot \mathcal{B}(A_1, A_2)^{-1}.\end{aligned}$$

These operations have the same expansion in first order, and the same scaling properties, and they are also involutive. It is easy to check that

- (1) $\mathcal{L}^{\text{fx}} \circ \mathcal{I}_C^{\text{fy}} = \mathcal{L}^{\text{fx}} \circ \mathcal{I}_L^{\text{fy}} = \mathcal{L}^{\text{fx}} \circ \mathcal{I}_R^{\text{fy}} = (\mathcal{L}^{\text{fx}})^{-1}$;
- (2) $\mathcal{D}^{\text{fx}} \circ \mathcal{I}_C^{\text{fy}} = \mathcal{D}^{\text{fx}} \circ \mathcal{I}_L^{\text{fy}} = \mathcal{D}^{\text{fx}} \circ \mathcal{I}_R^{\text{fy}} = -(\mathcal{D}^{\text{fx}})^{-1}$.

There are some variants given by

$$\begin{aligned}\mathcal{I}_L^{\text{y}}(A_1, A_2) &:= \mathcal{B}(A_1, A_2)^{-1} \cdot \mathcal{T}(A_1, A_2) \cdot \mathcal{L}^{\text{cx}}(A_1, A_2)^{-1}, \\ \mathcal{I}_R^{\text{y}}(A_1, A_2) &:= \mathcal{L}^{\text{cx}}(A_1, A_2)^{-1} \cdot \mathcal{T}(A_1, A_2) \cdot \mathcal{B}(A_1, A_2)^{-1}, \\ \mathcal{I}_L^{\text{cy}}(A_1, A_2) &:= \mathcal{B}(A_1, A_2)^{-1} \cdot \mathcal{T}(A_1, A_2) \cdot \mathcal{L}^{\text{x}}(A_1, A_2)^{-1}, \\ \mathcal{I}_R^{\text{cy}}(A_1, A_2) &:= \mathcal{L}^{\text{x}}(A_1, A_2)^{-1} \cdot \mathcal{T}(A_1, A_2) \cdot \mathcal{B}(A_1, A_2)^{-1}.\end{aligned}$$

They have the same first-order expansions as above; $\mathcal{I}_L^{\text{y}}, \mathcal{I}_R^{\text{y}}$ have scalings in $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4$, and $\mathcal{I}_L^{\text{cy}}, \mathcal{I}_R^{\text{cy}}$ have scalings in $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_5$; but these variants are not involutive. Also,

- (3) $\mathcal{I}_C^{\text{fy}}, \mathcal{I}_L^{\text{fy}}, \mathcal{I}_R^{\text{fy}}, \mathcal{I}_L^{\text{y}}, \mathcal{I}_R^{\text{y}}, \mathcal{I}_L^{\text{cy}}, \mathcal{I}_R^{\text{cy}}$ commute with \mathcal{T} and \mathcal{C} with respect to composition.

\mathcal{I}_C^{y} is reasonably nice but it lacks \hat{r}_4 scaling and hence affine invariance. What would be more interesting is to have a similar involutive and transposition invariant operation but with affine invariance properties (i. e. with orthogonal invariance and \hat{r}_3 and \hat{r}_4 scalings).

2. THE AXIAL EXTENSION PROCEDURE

2.1. Some of the latter examples are examples “axial extensions” which we explain as follows. Suppose that Ξ is a (sort of) conjugation-invariant FQ operation defined at least on monoaxial elements. Then we can set

$$\begin{aligned}\Xi^{\text{x}}(A_1, A_2) &:= \Xi(A_1 \mathcal{B}^{-1}, A_2 \mathcal{B}^{-1}) \cdot \mathcal{B} = \mathcal{B} \cdot \Xi(\mathcal{B}^{-1} A_1, \mathcal{B}^{-1} A_2) \\ &= \mathcal{B}^{1/2} \cdot \Xi(\mathcal{B}^{-1/2} A_1 \mathcal{B}^{-1/2}, \mathcal{B}^{-1/2} A_2 \mathcal{B}^{-1/2}) \cdot \mathcal{B}^{1/2}\end{aligned}$$

where $\mathcal{B} = \mathcal{B}(A_1, A_2)$. The equality of the various presentations is a consequence of conjugation invariance. For an FQ operation, being axial (extension) can be understood as having bivariance with respect to the biaxiality operation. Now, it is easy to see that this extension (or modification) process acts conservatively on monoaxial elements.

This axial extension procedure has a variant defined by

$$\Xi^{\text{cx}}(A_1, A_2) := \pm \Xi(A_1 \mathcal{A}_C, A_2 \mathcal{A}_C) \cdot \mathcal{A}_C^{-1} = \pm \mathcal{A}_C^{-1} \cdot \Xi(\mathcal{A}_C A_1, \mathcal{A}_C A_2),$$

where $\mathcal{A}_C = \mathcal{A}_C(A_1, A_2)$, with $\pm = +$ in the scalar and vectorial cases, and $\pm = -$ in the pseudoscalar case. Equivalence follows from naturality. This is more or the less the same as the original version: Formally, or analytically, if the environment is appropriate the following hold: If Ξ is scalar, then $\Xi^{\text{cx}} = \mathcal{A}_L \cdot \Xi^{\text{x}} = \Xi^{\text{x}} \cdot \mathcal{A}_R$; if Ξ is vectorial, then $\Xi^{\text{cx}} = \Xi^{\text{x}}$; if Ξ is pseudoscalar, then $\Xi^{\text{cx}} = -\mathcal{A}_L \cdot \Xi^{\text{x}} = -\Xi^{\text{x}} \cdot \mathcal{A}_R$. (It interchanges scalar and pseudoscalar.)

2.2. The axial extension procedure can be understood on the formal level as follows. What happens is that for perturbations $(A_1, A_2) = (Q_1 + R_1, Q_2 + R_2)$ of (Q_1, Q_2) we have decomposition

$$(Q_1 + R_1, Q_2 + R_2) = ((1 + \hat{r}_1 + \hat{r}_2 + \hat{r}_3 + \hat{r}_4 + \hat{r}_5 + \hat{r}_6 + \hat{r}_7 + \hat{r}_8)Q_1, \\ (1 + \hat{r}_1 - \hat{r}_2 + \hat{r}_3 - \hat{r}_4 - \hat{r}_5 + \hat{r}_6 + \hat{r}_7 - \hat{r}_8)Q_2).$$

Now, if (A_1, A_2) , and its axis is Q_1Q_2 , then according to what we have seen, monoaxiality makes (A_1, A_2) anticommute with Q_1Q_2 ; hence $\hat{r}_1 = \hat{r}_2 = \hat{r}_7 = \hat{r}_8 = 0$ in the decomposition. So,

$$(Q_1 + R_1, Q_2 + R_2) = ((1 + \hat{r}_3 + \hat{r}_4 + \hat{r}_5 + \hat{r}_6)Q_1, (1 + \hat{r}_3 - \hat{r}_4 - \hat{r}_5 + \hat{r}_6)Q_2).$$

Ultimately, one has a restricted formal FQ calculus for monoaxial operations, which is similar to the original one but with a restricted set of variables $\hat{r}_3, \hat{r}_4, \hat{r}_5, \hat{r}_6$. In this case $\mathcal{O}^{\text{fSy}}(A_1, A_2) = (Q_1, Q_2)$ if and only if $\hat{r}_6 = 0$. Hence, there is a similar discussion of conjugation-invariance leading to the eliminability of 6-indices (as opposed to 6, 7, 8-indices). One can also see that it can also be described by the (restricted version) of the 6-hyperscaling condition.

Notice that using the usual expansion formulas we cannot create non-monoaxial pairs from monoaxial pairs because an expression of $\hat{r}_3, \hat{r}_4, \hat{r}_5, \hat{r}_6$ still commutes with Q_1Q_2 . (“No escape from monoaxiality.”) Similarly, if one has a pseudoscalar Clifford conservative operation, then it must be multiplicatively conjugate to \mathcal{A}_C , hence by the reason above, it must be \mathcal{A}_C itself. Also notice that in the monoaxial calculus, multiplication by \mathcal{A}_C makes a bijective correspondence between scalar and pseudoscalar operations.

2.3. Theorem. *In terms of expansions regarding the mixed base, being an axial extension means that lower $\{1, 2, 7, 8\}$ -indices can be eliminated from $\hat{p}_{t_1, \dots, t_r}^{[s]}$; or, taking conjugation-invariance into account, it means the eliminability of $\{1, 2\}$ -indices, hence reduction to lower $\{3, 4, 5\}$ -indices.*

Proof. This follows from conservativity on monoaxial elements and using any of the extension formulas from 2.1. \square

In the Clifford conservative case, for an axial extension Ξ^x , in the scalar case $\hat{q}_1^{[0]} = 0$ and $\hat{q}_2^{[0]} = 2$; in the vectorial case $\hat{q}_1^{[1]} = \hat{q}_2^{[1]} = 1$; in the pseudoscalar case $\hat{q}_1^{[12]} = \hat{q}_2^{[12]} = 0$.

We can also interpret $\Xi^m := \Xi \circ \mathcal{M}_C$ as a kind of axial extension with $\hat{q}_1^{[s]} = \hat{q}_2^{[s]} = 0$. Or, we can consider

$$\Xi^y(A_1, A_2) := \mathcal{B}^{-1} \cdot \Xi(A_1\mathcal{B}^{-1}, A_2\mathcal{B}^{-1}) = \Xi(\mathcal{B}^{-1}A_1, \mathcal{B}^{-1}A_2) \cdot \mathcal{B}^{-1} \\ = \mathcal{B}^{-1/2} \cdot \Xi(\mathcal{B}^{-1/2}A_1\mathcal{B}^{-1/2}, \mathcal{B}^{-1/2}A_2\mathcal{B}^{-1/2}) \cdot \mathcal{B}^{-1/2}$$

as an axial extension. Then, for the extension Ξ^y of a Clifford conservative operation Ξ , in the scalar case $\hat{q}_1^{[0]} = 0$ and $\hat{q}_2^{[0]} = -2$; in the vectorial case $\hat{q}_1^{[1]} = \hat{q}_2^{[1]} = -1$; in the pseudoscalar case $\hat{q}_1^{[12]} = \hat{q}_2^{[12]} = 0$. (Theorem 2.3 still applies.)

It is easy to see that being an axial extension automatically induces compatibility to the central axis operation: If Ξ is a vectorial Clifford conservative FQ operation, then

- (1) $\mathcal{A}_X \circ \Xi^x = \mathcal{A}_X, \mathcal{B} \circ \Xi^x = \mathcal{B}$;
- (2) $\mathcal{A}_X \circ \Xi^m = \mathcal{A}_C, \mathcal{B} \circ \Xi^m = 1$;
- (3) $\mathcal{A}_X \circ \Xi^y = \mathcal{A}_{-X}, \mathcal{B} \circ \Xi^y = \mathcal{B}^{-1}$.

Furthermore, one can show that Ξ^x, Ξ^m, Ξ^y will automatically satisfy scaling invariances in \hat{r}_1 and \hat{r}_2 (for all types, not only in the vectorial case).

3. EXAMPLES OF HYPERSCALING

In [7] hyperscaling had some success in describing conjugation-invariance and bivariate. Furthermore, these conditions come in a great variety; hence one hopes that they might be somewhat interesting. On the other hand, in [7], it was also indicated that while these conditions have a quite complex behaviour, they are too strong to be very useful. Here we try to enlighten this situation. In this section we work in the formal environment, all FQ operations are meant in their formal restrictions.

3.1. Recall from [7] that the FQ operation Ξ satisfies the hyperscaling property of type (J, L, α, β) in variable \hat{r}_h , component $[s]$, if in its expansion relative to the mixed base, the “decay” identities

$$\begin{aligned}\hat{p}_h^{[s]} &= (\alpha + \beta)\hat{p}^{[s]} \\ \hat{p}_{h,j,\dots}^{[s]} &= \alpha\hat{p}_{j,\dots}^{[s]} - \frac{1}{2}J\hat{p}_{6*h*j,\dots}^{[s]} + (-\frac{1}{2} - L)\hat{p}_{h*j,\dots}^{[s]} \\ \hat{p}_{\dots,i,h}^{[s]} &= \frac{1}{2}J\hat{p}_{\dots,i*h*6}^{[s]} + (-\frac{1}{2} + L)\hat{p}_{\dots,i*h}^{[s]} + \beta\hat{p}_{\dots,i}^{[s]} \\ \hat{p}_{\dots,i,h,j,\dots}^{[s]} &= \frac{1}{2}J\hat{p}_{\dots,i*h*6,j,\dots}^{[s]} + (-\frac{1}{2} + L)\hat{p}_{\dots,i*h,j,\dots}^{[s]} - \frac{1}{2}J\hat{p}_{\dots,i,6*h*j,\dots}^{[s]} + (-\frac{1}{2} - L)\hat{p}_{\dots,i,h*j,\dots}^{[s]}\end{aligned}$$

hold. In what follows, this conditions will be abbreviated as $h[s](J, L, \alpha, \beta)$.

Similarly, the FQ operation Ξ satisfies character degeneracy with ± 1 in variable \hat{r}_h , component $[s]$, if in its expansion relative to the mixed base, the identities

$$\hat{p}_{\dots,i,\dots}^{[s]} = \pm \hat{p}_{\dots,i*h,\dots}^{[s]}$$

hold. (This is more general compared to [7], where character degeneracy was considered only in variable \hat{r}_6 .) The condition above will be abbreviated as $h[s]\langle +1 \rangle$ or $h[s]\langle -1 \rangle$.

It turns out, these conditions are surprisingly structured. We start with the $L = 0$ case. We will name some special types

$$\begin{array}{lll}\mathbf{1} = (1, 0, 1, 1), & \mathbf{2} = (1, 0, 1, -1), & \mathbf{3} = (1, 0, -1, 1), \\ \mathbf{4} = (1, 0, -1, -1), & \mathbf{5} = (-1, 0, 0, 0), & \mathbf{6} = (-1, 0, 0, 1), \\ \mathbf{7} = (-1, 0, 0, -1), & \mathbf{8} = (1, 0, 1, 0), & \mathbf{9} = (1, 0, -1, 0).\end{array}$$

They fit into the picture

$L = 0$	$\beta = 1$	$\beta = 0$	$\beta = -1$
$\alpha = 1, J = 1$	1	8	2
$\alpha = 0, J = -1$	6	5	7
$\alpha = -1, J = 1$	3	9	4

In this terminology conjugation-invariance can be described by

$$(6[0]\mathbf{2} \text{ or } 6[0]\mathbf{5}) \text{ and } 7[0]\mathbf{2} \text{ and } 8[0]\mathbf{5}$$

in part [0];

$$(6[1]\mathbf{6} \text{ or } 6[1]\mathbf{8}) \text{ and } 7[1]\mathbf{8} \text{ and } 8[1]\mathbf{6}$$

in part [1];

$$(6[2]\mathbf{6} \text{ or } 6[2]\mathbf{8}) \text{ and } 7[2]\mathbf{8} \text{ and } 8[2]\mathbf{7}$$

in part [2];

$$(6[12]\mathbf{2} \text{ or } 6[12]\mathbf{5}) \text{ and } 7[12]\mathbf{1} \text{ and } 8[12]\mathbf{5}$$

in part [12]. (We have some leverage in choosing parameters for \hat{r}_6 .)

3.2. Statement. (a) If Ξ is a conjugation-invariant Clifford conservative (pseudo)scalar FQ operation satisfying the hyperscaling property of type $(J, 0, \alpha, \beta)$ in variable \hat{r}_h ($h \in \{1, 2, 3, 4, 5\}$), component $[s]$ ($s \in \{0, 12\}$). Then, we claim, Ξ satisfies a hyperscaling property of type **1, 2, 3, 4** or **5** in variable \hat{r}_h , component $[s]$.

(b) If Ξ is a conjugation-invariant Clifford conservative vectorial FQ operation satisfying the hyperscaling property of type $(J, 0, \alpha, \beta)$ in variable \hat{r}_h ($h \in \{1, 2, 3, 4, 5\}$), component $[s]$ ($s \in \{1, 2\}$). Then, we, claim Ξ satisfies a hyperscaling property of type **5, 6, 7** or **9** in variable \hat{r}_h , component $[s]$.

(Remark: We do not claim that $(J, 0, \alpha, \beta)$ itself is one of **1, ..., 9**.)

We call the $(5 + 2 \times 4 + 5) \times 5 = 90$ possible hyperscaling conditions above as the principal hyperscaling conditions. The following statement refines the situation. It turns out that if we impose a principal hyperscaling condition $h[s]X$, then it does not only allow to eliminate h from the lower indices of $\hat{p}_{\iota_1, \dots, \iota_r}^{[s]}$ (beyond 6, 7, 8) but implies further rules. Ultimately, this will allow a reducing index set $I_e \subsetneq \{1, \dots, 5\} \setminus \{h\}$ such that the expansion coefficients will depend on $\hat{p}_{\iota_1, \dots, \iota_r}^{[s]}$ ($\iota_1, \dots, \iota_r \in I$) which can be prescribed arbitrarily (with $\hat{p}^{[s]} = 1$ in the Clifford conservative case). We term these as exact reduction sets I_e . When we pair hyperscaling properties with, say orthogonal invariances, we can obtain index sets I_{ne} such that lower indices can be reduced to be from I_{ne} but subject to further conditions, i. e. free prescribability does not hold.

In the following statements we deal with Clifford conservative operations. $\hat{\mathbf{P}}_{1\dots 5}^{[s]}$ means $\hat{\mathbf{P}}_1^{[s]}$ restricted to its first 5 entries.

3.3. Statement. For scalar FQ operations Ξ , the consistent constellations of principal hyperscaling properties are as follows:

- (1) $1[0]1 \Leftrightarrow 4[0]1 \rightsquigarrow I_e = \{2, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [2 \quad \hat{q}_2 \quad \hat{q}_3 \quad 2 \quad \hat{q}_5]$
- (2) $1[0]2 \Leftrightarrow 4[0]2 \rightsquigarrow I_e = \{2, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \quad \hat{q}_2 \quad \hat{q}_3 \quad 0 \quad \hat{q}_5]$
- (3) $1[0]3 \Leftrightarrow 4[0]3 \rightsquigarrow I_e = \{2, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \quad \hat{q}_2 \quad \hat{q}_3 \quad 0 \quad \hat{q}_5]$
- (4) $1[0]4 \Leftrightarrow 4[0]4 \rightsquigarrow I_e = \{2, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [-2 \quad \hat{q}_2 \quad \hat{q}_3 \quad -2 \quad \hat{q}_5]$
- (5) $1[0]5 \Leftrightarrow 3[0]5 \rightsquigarrow I_e = \{2, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \quad \hat{q}_2 \quad 0 \quad \hat{q}_4 \quad \hat{q}_5]$
- (6) $2[0]1 \Leftrightarrow 3[0]1 \rightsquigarrow I_e = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_1 \quad 2 \quad 2 \quad \hat{q}_4 \quad \hat{q}_5]$
- (7) $2[0]2 \Leftrightarrow 3[0]2 \rightsquigarrow I_e = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_1 \quad 0 \quad 0 \quad \hat{q}_4 \quad \hat{q}_5]$
- (8) $2[0]3 \Leftrightarrow 3[0]3 \rightsquigarrow I_e = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_1 \quad 0 \quad 0 \quad \hat{q}_4 \quad \hat{q}_5]$
- (9) $2[0]4 \Leftrightarrow 3[0]4 \rightsquigarrow I_e = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_1 \quad -2 \quad -2 \quad \hat{q}_4 \quad \hat{q}_5]$
- (10) $2[0]5 \Leftrightarrow 4[0]5 \rightsquigarrow I_e = \{1, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_1 \quad 0 \quad \hat{q}_3 \quad 0 \quad \hat{q}_5]$
- (11) $5[0]2 \Leftrightarrow 1[0]\langle +1 \rangle \rightsquigarrow I_e = \{3, 4\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_3 \quad \hat{q}_4 \quad \hat{q}_3 \quad \hat{q}_4 \quad 0]$
- (12) $5[0]3 \Leftrightarrow 1[0]\langle -1 \rangle \rightsquigarrow I_e = \{3, 4\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [-\hat{q}_3 \quad -\hat{q}_4 \quad \hat{q}_3 \quad \hat{q}_4 \quad 0]$
- (13) $5[0]5 \Leftrightarrow 2[0]\langle +1 \rangle \rightsquigarrow I_e = \{3, 4\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_4 \quad \hat{q}_3 \quad \hat{q}_3 \quad \hat{q}_4 \quad 0]$
- (14) $1, 4[0]2 \& 2, 4[0]5 \Leftrightarrow 5[0]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \quad 0 \quad \hat{q}_5 \quad 0 \quad \hat{q}_5]$
- (15) $1, 4[0]3 \& 2, 4[0]5 \Leftrightarrow 5[0]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \quad 0 \quad -\hat{q}_5 \quad 0 \quad \hat{q}_5]$
- (16) $1, 4[0]2 \& 1, 3[0]5 \Leftrightarrow 8[0]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \quad \hat{q}_5 \quad 0 \quad 0 \quad \hat{q}_5]$
- (17) $1, 4[0]3 \& 1, 3[0]5 \Leftrightarrow 8[0]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \quad -\hat{q}_5 \quad 0 \quad 0 \quad \hat{q}_5]$
- (18) $1, 3[0]5 \& 2, 3[0]2 \Leftrightarrow 6[0]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \quad 0 \quad 0 \quad \hat{q}_5 \quad \hat{q}_5]$
- (19) $1, 3[0]5 \& 2, 3[0]3 \Leftrightarrow 6[0]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \quad 0 \quad 0 \quad -\hat{q}_5 \quad \hat{q}_5]$

- (20) $2, 3[0]2 \& 2, 4[0]5 \Leftrightarrow 7[0]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_5 \ 0 \ 0 \ 0 \ \hat{q}_5]$
(21) $2, 3[0]3 \& 2, 4[0]5 \Leftrightarrow 7[0]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [-\hat{q}_5 \ 0 \ 0 \ 0 \ \hat{q}_5]$
(22) $1, 3[0]5 \& 5[0]2 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ \hat{q}_4 \ 0 \ \hat{q}_4 \ 0]$
(23) $1, 3[0]5 \& 5[0]3 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ -\hat{q}_4 \ 0 \ \hat{q}_4 \ 0]$
(24) $2, 3[0]1 \& 5[0]5 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_4 \ 2 \ 2 \ \hat{q}_4 \ 0]$
(25) $2, 3[0]2 \& 5[0]5 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_4 \ 0 \ 0 \ \hat{q}_4 \ 0]$
(26) $2, 3[0]3 \& 5[0]5 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_4 \ 0 \ 0 \ \hat{q}_4 \ 0]$
(27) $2, 3[0]4 \& 5[0]5 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_4 \ -2 \ -2 \ \hat{q}_4 \ 0]$
(28) $1, 4[0]1 \& 5[0]5 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [2 \ \hat{q}_3 \ \hat{q}_3 \ 2 \ 0]$
(29) $1, 4[0]2 \& 5[0]5 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ \hat{q}_3 \ \hat{q}_3 \ 0 \ 0]$
(30) $1, 4[0]3 \& 5[0]5 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ \hat{q}_3 \ \hat{q}_3 \ 0 \ 0]$
(31) $1, 4[0]4 \& 5[0]5 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [-2 \ \hat{q}_3 \ \hat{q}_3 \ -2 \ 0]$
(32) $2, 4[0]5 \& 5[0]2 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_3 \ 0 \ \hat{q}_3 \ 0 \ 0]$
(33) $2, 4[0]5 \& 5[0]3 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [-\hat{q}_3 \ 0 \ \hat{q}_3 \ 0 \ 0]$
(34) $5[0]2 \& 5[0]5 \Leftrightarrow 4[0]\langle +1 \rangle \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [\hat{q}_3 \ \hat{q}_3 \ \hat{q}_3 \ \hat{q}_3 \ 0]$
(35) $5[0]3 \& 5[0]5 \Leftrightarrow 4[0]\langle -1 \rangle \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [-\hat{q}_3 \ \hat{q}_3 \ \hat{q}_3 \ -\hat{q}_3 \ 0]$
(36) $1, 4[0]1 \& 2, 3[0]1 \& 5[0]2 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [2 \ 2 \ 2 \ 2 \ 0]$
and in this case $\Xi(A_1, A_2) = -A_1^2$
(37) $1, 4[0]4 \& 2, 3[0]1 \& 5[0]3 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [-2 \ 2 \ 2 \ -2 \ 0]$
and in this case $\Xi(A_1, A_2) = -A_2^2$
(38) $1, 4[0]1 \& 2, 3[0]4 \& 5[0]3 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [2 \ -2 \ -2 \ 2 \ 0]$
and in this case $\Xi(A_1, A_2) = -A_2^{-2}$
(39) $1, 4[0]4 \& 2, 3[0]4 \& 5[0]2 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [-2 \ -2 \ -2 \ -2 \ 0]$
and in this case $\Xi(A_1, A_2) = -A_1^{-2}$
(40) $1, 2, 3, 4, 5[0]2 \& 1, 2, 3, 4, 5[0]3 \& 1, 2, 3, 4, 5[0]5 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ 0 \ 0 \ 0 \ 0]$
and in this case $\Xi(A_1, A_2) = 1$.

Remark: Conditions $5[0]1, 5[0]4, 2[0]\langle -1 \rangle, 3[0]\langle -1 \rangle$ are inconsistent; $3[0]\langle +1 \rangle$ is trivial.

We can say that conditions (1–13) are the primary conditions, (14–35) are composite conditions, (36–40) are extremal conditions.

3.4. Statement. *For orthogonal invariant scalar FQ operations Ξ , the consistent constellations of principal hyperscaling properties are as follows:*

- (1) $1[0]5 \Leftrightarrow 3[0]5 \rightsquigarrow I_{ne} = \{2, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ \hat{q}_2 \ 0 \ 0 \ 0]$
(2) $2[0]1 \Leftrightarrow 3[0]1 \rightsquigarrow I_{ne} = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ 2 \ 2 \ 0 \ 0]$
(3) $2[0]2 \Leftrightarrow 3[0]2 \rightsquigarrow I_{ne} = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ 0 \ 0 \ 0 \ 0]$
(4) $2[0]3 \Leftrightarrow 3[0]3 \rightsquigarrow I_{ne} = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ 0 \ 0 \ 0 \ 0]$
(5) $2[0]4 \Leftrightarrow 3[0]4 \rightsquigarrow I_{ne} = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ -2 \ -2 \ 0 \ 0]$
(6) $2[0]5 \Leftrightarrow 4[0]5 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ 0 \ \hat{q}_3 \ 0 \ 0]$
(7) $5[0]5 \Leftrightarrow 2[0]\langle +1 \rangle \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ \hat{q}_3 \ \hat{q}_3 \ 0 \ 0]$
(8) $2, 3[0]1 \& 5[0]5 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = [0 \ 2 \ 2 \ 0 \ 0]$
and in this case $\Xi(A_1, A_2) = \mathcal{L}(A_1, A_2) \equiv -\frac{1}{2}(A_1^2 + A_2^2)$

$$(9) \quad 2, 3[0]4 \& 5[0]5 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = \begin{bmatrix} 0 & -2 & -2 & 0 & 0 \end{bmatrix}$$

and in this case $\Xi(A_1, A_2) = \mathcal{L}(A_1, A_2)^{-1} \equiv -\left(\frac{1}{2}(A_1^2 + A_2^2)\right)^{-1}$

$$(10) \quad 1, 2, 3, 4, 5[0]2 \& 1, 2, 3, 4, 5[0]3 \& 1, 2, 3, 4, 5[0]5 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[0]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and in this case $\Xi(A_1, A_2) = 1$.

We can say that conditions (1–7) are the primary conditions, (8–10) are extremal conditions.

3.5. Statement. For orthogonal invariant scalar FQ operations,

- an example for 3.4(1) is $\Xi(A_1, A_2) = \mathcal{B}(A_1, A_2)$ with values $\hat{q}_2 = 2$, $\hat{\mathbf{P}}_{1\dots 5}^{[0]} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \end{bmatrix}$;
- an example for 3.4(1) is $\Xi(A_1, A_2) = \mathcal{B}^{-1}(A_1, A_2)$ with values $\hat{q}_2 = -2$, $\hat{\mathbf{P}}_{1\dots 5}^{[0]} = \begin{bmatrix} 0 & -2 & 0 & 0 & 0 \end{bmatrix}$;
- examples for 3.4(2) are $\Xi = \mathcal{L}^{\text{fx}}, \mathcal{L}^{\text{x}}, \mathcal{L}^{\text{cx}}$;
- examples for 3.4(5) are $\Xi = (\mathcal{L}^{\text{fx}})^{-1}, (\mathcal{L}^{\text{x}})^{-1}, (\mathcal{L}^{\text{cx}})^{-1}$;
- an example for 3.4(6) is $\Xi(A_1, A_2) = |\mathcal{D}(A_1, A_2)|$ with values $\hat{q}_3 = 2$, $\hat{\mathbf{P}}_{1\dots 5}^{[0]} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \end{bmatrix}$;
- an example for 3.4(6) is $\Xi(A_1, A_2) = |\mathcal{D}(A_1, A_2)|^{-1}$ with values $\hat{q}_3 = -2$, $\hat{\mathbf{P}}_{1\dots 5}^{[0]} = \begin{bmatrix} 0 & 0 & -2 & 0 & 0 \end{bmatrix}$.

3.6. Statement. For vectorial FQ operations Ξ , which can be assumed to be symmetric, the consistent constellations of principal hyperscaling properties in component [1] are as follows:

- (1) $1[1]6 \Leftrightarrow 3[1]6 \rightsquigarrow I_e = \{2, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & \hat{q}_2 & 1 & \hat{q}_4 & \hat{q}_5 \end{bmatrix}$
- (2) $1[1]7 \Leftrightarrow 3[1]7 \rightsquigarrow I_e = \{2, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & \hat{q}_2 & -1 & \hat{q}_4 & \hat{q}_5 \end{bmatrix}$
- (3) $1[1]8 \Leftrightarrow 4[1]8 \rightsquigarrow I_e = \{2, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & \hat{q}_2 & \hat{q}_3 & 1 & \hat{q}_5 \end{bmatrix}$
- (4) $1[1]9 \Leftrightarrow 4[1]9 \rightsquigarrow I_e = \{2, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & \hat{q}_2 & \hat{q}_3 & -1 & \hat{q}_5 \end{bmatrix}$
- (5) $2[1]6 \Leftrightarrow 4[1]6 \rightsquigarrow I_e = \{1, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} \hat{q}_1 & 1 & \hat{q}_3 & 1 & \hat{q}_5 \end{bmatrix}$
- (6) $2[1]7 \Leftrightarrow 4[1]7 \rightsquigarrow I_e = \{1, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} \hat{q}_1 & -1 & \hat{q}_3 & -1 & \hat{q}_5 \end{bmatrix}$
- (7) $2[1]8 \Leftrightarrow 3[1]8 \rightsquigarrow I_e = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} \hat{q}_1 & 1 & 1 & \hat{q}_4 & \hat{q}_5 \end{bmatrix}$
- (8) $2[1]9 \Leftrightarrow 3[1]9 \rightsquigarrow I_e = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} \hat{q}_1 & -1 & -1 & \hat{q}_4 & \hat{q}_5 \end{bmatrix}$
- (9) $5[1]6 \Leftrightarrow 2[1]\langle +1 \rangle \rightsquigarrow I_e = \{3, 4\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} \hat{q}_4 & \hat{q}_3 & \hat{q}_3 & \hat{q}_4 & 1 \end{bmatrix}$
- (10) $5[1]8 \Leftrightarrow 1[1]\langle +1 \rangle \rightsquigarrow I_e = \{3, 4\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} \hat{q}_3 & \hat{q}_4 & \hat{q}_3 & \hat{q}_4 & 1 \end{bmatrix}$
- (11) $1, 3[1]6 \& 2, 3[1]8 \Leftrightarrow 6[1]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & 1 & 1 & \hat{q}_5 & \hat{q}_5 \end{bmatrix}$
- (12) $1, 3[1]7 \& 2, 3[1]9 \Leftrightarrow 6[1]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & -1 & -1 & -\hat{q}_5 & \hat{q}_5 \end{bmatrix}$
- (13) $2, 4[1]6 \& 2, 3[1]8 \Leftrightarrow 7[1]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} \hat{q}_5 & 1 & 1 & 1 & \hat{q}_5 \end{bmatrix}$
- (14) $2, 4[1]7 \& 2, 3[1]9 \Leftrightarrow 7[1]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -\hat{q}_5 & -1 & -1 & -1 & \hat{q}_5 \end{bmatrix}$
- (15) $1, 3[1]6 \& 1, 4[1]8 \Leftrightarrow 8[1]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & \hat{q}_5 & 1 & 1 & \hat{q}_5 \end{bmatrix}$
- (16) $1, 3[1]7 \& 1, 4[1]9 \Leftrightarrow 8[1]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & -\hat{q}_5 & -1 & -1 & \hat{q}_5 \end{bmatrix}$
- (17) $1, 4[1]8 \& 2, 4[1]6 \Leftrightarrow 5[1]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & 1 & \hat{q}_5 & 1 & \hat{q}_5 \end{bmatrix}$
- (18) $1, 4[1]9 \& 2, 4[1]7 \Leftrightarrow 5[1]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & -1 & -\hat{q}_5 & -1 & \hat{q}_5 \end{bmatrix}$
- (19) $1, 3[1]6 \& 5[1]8 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & \hat{q}_4 & 1 & \hat{q}_4 & 1 \end{bmatrix}$
- (20) $1, 3[1]7 \& 5[1]8 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & \hat{q}_4 & -1 & \hat{q}_4 & 1 \end{bmatrix}$

- (21) $2, 3[1]8 \& 5[1]6 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [\hat{q}_4 \ 1 \ 1 \ \hat{q}_4 \ 1]$
(22) $2, 3[1]9 \& 5[1]6 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [\hat{q}_4 \ -1 \ -1 \ \hat{q}_4 \ 1]$
(23) $1, 4[1]8 \& 5[1]6 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ \hat{q}_3 \ \hat{q}_3 \ 1 \ 1]$
(24) $1, 4[1]9 \& 5[1]6 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [-1 \ \hat{q}_3 \ \hat{q}_3 \ -1 \ 1]$
(25) $2, 4[1]6 \& 5[1]8 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [\hat{q}_3 \ 1 \ \hat{q}_3 \ 1 \ 1]$
(26) $2, 4[1]7 \& 5[1]8 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [-\hat{q}_3 \ -1 \ -\hat{q}_3 \ -1 \ 1]$
(27) $5[1]6 \& 5[1]8 \Leftrightarrow 4[1]\langle +1 \rangle \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [\hat{q}_3 \ \hat{q}_3 \ \hat{q}_3 \ \hat{q}_3 \ 1]$
(28) $1, 2, 3, 4, 5[1]6 \& 1, 2, 3, 4, 5[1]8 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ 1 \ 1 \ 1 \ 1]$
and in this case $\Xi(A_1, A_2) = \text{Id}(A_1, A_2) = (A_1, A_2)$
(29) $1, 2, 3, 4[1]7 \& 1, 2, 3, 4[1]9 \& 5[1]6 \& 5[1]8 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [-1 \ -1 \ -1 \ -1 \ 1]$
and in this case $\Xi(A_1, A_2) = (-A_1^{-1}, -A_2^{-1})$

Remark: Conditions $5[1]7, 5[1]9, 1[1]\langle -1 \rangle, 2[1]\langle -1 \rangle, 3[1]\langle -1 \rangle, 4[1]\langle -1 \rangle$ are inconsistent; $3[1]\langle +1 \rangle$ is trivial.

We can say that conditions (1–10) are the primary conditions, (11–27) are composite conditions, (28–29) are extremal conditions.

3.7. Statement. *For orthogonal-invariant vectorial FQ operations Ξ (which are necessarily symmetric), the consistent constellations of principal hyperscaling properties in component [1] are as follows:*

- (1) $1[1]6 \Leftrightarrow 3[1]6 \rightsquigarrow I_{\text{ne}} = \{2, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ \hat{q}_2 \ 1 \ \hat{q}_5 \ \hat{q}_5]$
(2) $1[1]7 \Leftrightarrow 3[1]7 \rightsquigarrow I_{\text{ne}} = \{2, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [-1 \ \hat{q}_2 \ -1 \ \hat{q}_5 \ \hat{q}_5]$
(3) $2[1]8 \Leftrightarrow 3[1]8 \rightsquigarrow I_{\text{ne}} = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [\hat{q}_1 \ 1 \ 1 \ \hat{q}_5 \ \hat{q}_5]$
(4) $2[1]9 \Leftrightarrow 3[1]9 \rightsquigarrow I_{\text{ne}} = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [\hat{q}_1 \ -1 \ -1 \ \hat{q}_5 \ \hat{q}_5]$
(5) $2[1]6 \Leftrightarrow 4[1]6 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ 1 \ \hat{q}_3 \ 1 \ 1]$
(6) $2[1]7 \Leftrightarrow 4[1]7 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ -1 \ \hat{q}_3 \ -1 \ -1]$
(7) $1[1]8 \Leftrightarrow 4[1]8 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ 1 \ \hat{q}_3 \ 1 \ 1]$
(8) $1[1]9 \Leftrightarrow 4[1]9 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [-1 \ 1 \ \hat{q}_3 \ -1 \ -1]$
(9) $5[1]6 \Leftrightarrow 2[1]\langle +1 \rangle \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ \hat{q}_3 \ \hat{q}_3 \ 1 \ 1]$
(10) $5[1]8 \Leftrightarrow 1[1]\langle +1 \rangle \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [\hat{q}_3 \ 1 \ \hat{q}_3 \ 1 \ 1]$
(11) $1, 3[1]6 \& 2, 3[1]8 \Leftrightarrow 6[1]\langle +1 \rangle \rightsquigarrow I_{\text{ne}} = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ 1 \ 1 \ \hat{q}_5 \ \hat{q}_5]$
and in this case $\Xi(A_1, A_2) = \hat{q}_5 \cdot (A_1, A_2) + (1 - \hat{q}_5) \cdot \mathcal{O}^{\text{fSy}}(A_1, A_2)$
(12) $1, 3[1]7 \& 2, 3[1]9 \Leftrightarrow 6[1]\langle -1 \rangle \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [-1 \ -1 \ -1 \ 0 \ 0]$
and in this case $\Xi(A_1, A_2) = \mathcal{O}^{\text{afSy}}(A_1, A_2)$
(13) $2, 4[1]7 \& 2, 3[1]9 \Leftrightarrow 7[1]\langle -1 \rangle \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ -1 \ -1 \ -1 \ -1]$
and in this case $\Xi(A_1, A_2) = \mathcal{D}(A_1, A_2)^{-1} \cdot (A_2, -A_1)$
(14) $1, 3[1]7 \& 1, 4[1]9 \Leftrightarrow 8[1]\langle -1 \rangle \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [-1 \ 1 \ -1 \ -1 \ -1]$
and in this case $\Xi(A_1, A_2) = (-A_2, A_1) \cdot \mathcal{D}(A_1, A_2)^{-1}$
(15) $1, 3[1]7 \& 5[1]8 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [-1 \ 1 \ -1 \ 1 \ 1]$
and in this case $\Xi(A_1, A_2) = (A_1, A_2) \cdot \mathcal{L}(A_1, A_2)^{-1}$
(16) $2, 3[1]9 \& 5[1]6 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ -1 \ -1 \ 1 \ 1]$
and in this case $\Xi(A_1, A_2) = \mathcal{L}(A_1, A_2)^{-1} \cdot (A_1, A_2)$
(17) $1, 2, 3, 4, 5[1]6 \& 1, 2, 3, 4, 5[1]8 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = [1 \ 1 \ 1 \ 1 \ 1]$

and in this case $\Xi(A_1, A_2) = \text{Id}(A_1, A_2) = (A_1, A_2)$

Moreover, if Ξ is also transposition invariant, then it is $\hat{q}_5 \cdot \text{Id} + (1 - \hat{q}_5) \cdot \mathcal{O}^{\text{fSy}}$ or $\mathcal{O}^{\text{afSy}}$.

We can say that conditions (1–10) are the primary conditions, (11) is a composite condition, (12–17) are extremal conditions.

3.8. Statement. For orthogonal invariant vectorial FQ operations,

- an example for 3.7(1) is $\Xi(A_1, A_2) = \mathcal{M}_R(A_1, A_2) \equiv \mathcal{B}(A_1, A_2)^{-1} \cdot (A_1, A_2)$ with $\hat{q}_2 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \end{bmatrix}$;
- an example for 3.7(3) is $\Xi(A_1, A_2) = \mathcal{M}_L(A_1, A_2) \equiv (A_1, A_2) \cdot \mathcal{B}(A_1, A_2)^{-1}$ with $\hat{q}_1 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & 1 & 1 & 1 & 1 \end{bmatrix}$;
- an example for 3.7(1) is $\Xi(A_1, A_2) = \mathcal{T}_{RR}(A_1, A_2) := \mathcal{A}_R(A_1, A_2) \cdot (-A_2, A_1)$ with $\hat{q}_2 = -3, \hat{q}_5 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & -3 & 1 & -1 & -1 \end{bmatrix}$;
- an example for 3.7(3) is $\Xi(A_1, A_2) = \mathcal{T}_{LL}(A_1, A_2) := (A_2, -A_1) \cdot \mathcal{A}_L(A_1, A_2)$ with $\hat{q}_1 = -3, \hat{q}_5 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -3 & 1 & 1 & -1 & -1 \end{bmatrix}$;
- an example for 3.7(1) is $\Xi(A_1, A_2) = \mathcal{T}_R(A_1, A_2) := \mathcal{A}_C(A_1, A_2) \cdot (-A_2, A_1)$ with $\hat{q}_2 = -1, \hat{q}_5 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix}$;
- an example for 3.7(3) is $\Xi(A_1, A_2) = \mathcal{T}_L(A_1, A_2) := (A_2, -A_1) \cdot \mathcal{A}_C(A_1, A_2)$ with $\hat{q}_1 = -1, \hat{q}_5 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & 1 & 1 & -1 & -1 \end{bmatrix}$;
- examples for 3.7(2) are $\Xi(A_1, A_2) = (A_1, A_2) \cdot \mathcal{L}^{\text{fx}}(A_1, A_2)^{-1}$,
 $\Xi(A_1, A_2) = (A_1, A_2) \cdot \mathcal{L}^{\text{x}}(A_1, A_2)^{-1}$, $\Xi(A_1, A_2) = (A_1, A_2) \cdot \mathcal{L}^{\text{cx}}(A_1, A_2)^{-1}$ with
 $\hat{q}_2 = 1, \hat{q}_5 = 1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & 1 & -1 & 1 & 1 \end{bmatrix}$;
- examples for 3.7(4) are $\Xi(A_1, A_2) = \mathcal{L}^{\text{fx}}(A_1, A_2)^{-1} \cdot (A_1, A_2)$,
 $\Xi(A_1, A_2) = \mathcal{L}^{\text{x}}(A_1, A_2)^{-1} \cdot (A_1, A_2)$, $\Xi(A_1, A_2) = \mathcal{L}^{\text{cx}}(A_1, A_2)^{-1} \cdot (A_1, A_2)$ with
 $\hat{q}_1 = 1, \hat{q}_5 = 1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \end{bmatrix}$;
- examples for 3.7(2) are $\Xi = \mathcal{I}_L^{\text{fy}}, \mathcal{I}_L^{\text{y}}, \mathcal{I}_L^{\text{cy}}$
with $\hat{q}_2 = -1, \hat{q}_5 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \end{bmatrix}$;
- examples for 3.7(4) are $\Xi = \mathcal{I}_R^{\text{fy}}, \mathcal{I}_R^{\text{y}}, \mathcal{I}_R^{\text{cy}}$
with $\hat{q}_1 = -1, \hat{q}_5 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \end{bmatrix}$;
- examples for 3.7(2) are $\Xi(A_1, A_2) = (-A_2, A_1) \cdot \mathcal{D}^{\text{fx}}(A_1, A_2)^{-1}$,
 $\Xi(A_1, A_2) = (-A_2, A_1) \cdot \mathcal{D}^{\text{x}}(A_1, A_2)^{-1}$, $\Xi(A_1, A_2) = (-A_2, A_1) \cdot \mathcal{D}^{\text{cx}}(A_1, A_2)^{-1}$
with $\hat{q}_2 = 1, \hat{q}_5 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} -1 & 1 & -1 & -1 & -1 \end{bmatrix}$;
- examples for 3.7(4) are $\Xi(A_1, A_2) = \mathcal{D}^{\text{fx}}(A_1, A_2)^{-1} \cdot (A_2, -A_1)$,
 $\Xi(A_1, A_2) = \mathcal{D}^{\text{x}}(A_1, A_2)^{-1} \cdot (A_2, -A_1)$, $\Xi(A_1, A_2) = \mathcal{D}^{\text{cx}}(A_1, A_2)^{-1} \cdot (A_2, -A_1)$
with $\hat{q}_1 = 1, \hat{q}_5 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \end{bmatrix}$;
- an example for 3.7(5) is $\Xi(A_1, A_2) = \mathcal{F}_R(A_1, A_2) := |\mathcal{D}(A_1, A_2)|^{-1} \cdot (A_1, A_2)$ with
 $\hat{q}_3 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \end{bmatrix}$;
- an example for 3.7(7) is $\Xi(A_1, A_2) = \mathcal{F}_L(A_1, A_2) := (A_1, A_2) \cdot |\mathcal{D}(A_1, A_2)|^{-1}$ with
 $\hat{q}_3 = -1, \hat{\mathbf{P}}_{1\dots 5}^{[1]} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \end{bmatrix}$.

3.9. Statement. For pseudoscalar FQ operations Ξ , the consistent constellations of principal hyperscaling properties are as follows:

- (1) $1[12]\mathbf{1} \Leftrightarrow 4[12]\mathbf{2} \rightsquigarrow I_e = \{2, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = \begin{bmatrix} 2 & \hat{q}_2 & \hat{q}_3 & 0 & \hat{q}_5 \end{bmatrix}$
- (2) $1[12]\mathbf{2} \Leftrightarrow 4[12]\mathbf{1} \rightsquigarrow I_e = \{2, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = \begin{bmatrix} 0 & \hat{q}_2 & \hat{q}_3 & 2 & \hat{q}_5 \end{bmatrix}$
- (3) $1[12]\mathbf{3} \Leftrightarrow 4[12]\mathbf{4} \rightsquigarrow I_e = \{2, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = \begin{bmatrix} 0 & \hat{q}_2 & \hat{q}_3 & -2 & \hat{q}_5 \end{bmatrix}$

- (4) $1[12]4 \Leftrightarrow 4[12]3 \rightsquigarrow I_e = \{2, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [-2 \ \hat{q}_2 \ \hat{q}_3 \ 0 \ \hat{q}_5]$
- (5) $1[12]5 \Leftrightarrow 3[12]5 \rightsquigarrow I_e = \{2, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ \hat{q}_2 \ 0 \ \hat{q}_4 \ \hat{q}_5]$
- (6) $2[12]1 \Leftrightarrow 3[12]2 \rightsquigarrow I_e = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [\hat{q}_1 \ 2 \ 0 \ \hat{q}_4 \ \hat{q}_5]$
- (7) $2[12]2 \Leftrightarrow 3[12]1 \rightsquigarrow I_e = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [\hat{q}_1 \ 0 \ 2 \ \hat{q}_4 \ \hat{q}_5]$
- (8) $2[12]3 \Leftrightarrow 3[12]4 \rightsquigarrow I_e = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [\hat{q}_1 \ 0 \ -2 \ \hat{q}_4 \ \hat{q}_5]$
- (9) $2[12]4 \Leftrightarrow 3[12]3 \rightsquigarrow I_e = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [\hat{q}_1 \ -2 \ 0 \ \hat{q}_4 \ \hat{q}_5]$
- (10) $2[12]5 \Leftrightarrow 4[12]5 \rightsquigarrow I_e = \{1, 3, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [\hat{q}_1 \ 0 \ \hat{q}_3 \ 0 \ \hat{q}_5]$
- (11) $5[12]1 \Leftrightarrow 1[12]\langle +1 \rangle \rightsquigarrow I_e = \{3, 4\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [\hat{q}_3 \ \hat{q}_4 \ \hat{q}_3 \ \hat{q}_4 \ 0]$
- (12) $5[12]4 \Leftrightarrow 1[12]\langle -1 \rangle \rightsquigarrow I_e = \{3, 4\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [-\hat{q}_3 \ -\hat{q}_4 \ \hat{q}_3 \ \hat{q}_4 \ 0]$
- (13) $1[12]1, 4[12]2 \& 2[12]5, 4[12]5 \Leftrightarrow 5[12]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [2 \ 0 \ \hat{q}_5 \ 0 \ \hat{q}_5]$
- (14) $1[12]4, 4[12]3 \& 2[12]5, 4[12]5 \Leftrightarrow 5[12]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [-2 \ 0 \ -\hat{q}_5 \ 0 \ \hat{q}_5]$
- (15) $1[12]5, 3[12]5 \& 2[12]1, 3[12]2 \Leftrightarrow 6[12]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 2 \ 0 \ \hat{q}_5 \ \hat{q}_5]$
- (16) $1[12]5, 3[12]5 \& 2[12]4, 3[12]3 \Leftrightarrow 6[12]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ -2 \ 0 \ -\hat{q}_5 \ \hat{q}_5]$
- (17) $2[12]2, 3[12]1 \& 2[12]5, 4[12]5 \Leftrightarrow 7[12]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [\hat{q}_5 \ 0 \ 2 \ 0 \ \hat{q}_5]$
- (18) $2[12]3, 3[12]4 \& 2[12]5, 4[12]5 \Leftrightarrow 7[12]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [-\hat{q}_5 \ 0 \ -2 \ 0 \ \hat{q}_5]$
- (19) $1[12]2, 4[12]1 \& 1[12]5, 3[12]5 \Leftrightarrow 8[12]\langle +1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ \hat{q}_5 \ 0 \ 2 \ \hat{q}_5]$
- (20) $1[12]3, 4[12]4 \& 1[12]5, 3[12]5 \Leftrightarrow 8[12]\langle -1 \rangle \rightsquigarrow I_e = \{5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ -\hat{q}_5 \ 0 \ -2 \ \hat{q}_5]$
- (21) $1[12]5, 3[12]5 \& 5[12]1 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ \hat{q}_4 \ 0 \ \hat{q}_4 \ 2]$
- (22) $1[12]5, 3[12]5 \& 5[12]4 \rightsquigarrow I_e = \{4\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ -\hat{q}_4 \ 0 \ \hat{q}_4 \ -2]$
- (23) $2[12]5, 4[12]5 \& 5[12]1 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [\hat{q}_3 \ 0 \ \hat{q}_3 \ 0 \ 2]$
- (24) $2[12]5, 4[12]5 \& 5[12]4 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [-\hat{q}_3 \ 0 \ \hat{q}_3 \ 0 \ -2]$
- (25) $1[12]1, 4[12]2 \& 2[12]2, 3[12]1 \& 5[12]1 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [2 \ 0 \ 2 \ 0 \ 2]$
and in this case $\Xi(A_1, A_2) = A_1 A_2$
- (26) $1[12]1, 4[12]2 \& 2[12]3, 3[12]4 \& 5[12]4 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [2 \ 0 \ -2 \ 0 \ -2]$
and in this case $\Xi(A_1, A_2) = -A_2^{-1} A_1^{-1}$
- (27) $1[12]4, 4[12]3 \& 2[12]2, 3[12]1 \& 5[12]4 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [-2 \ 0 \ 2 \ 0 \ -2]$
and in this case $\Xi(A_1, A_2) = -A_2 A_1$
- (28) $1[12]4, 4[12]3 \& 2[12]3, 3[12]4 \& 5[12]1 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [-2 \ 0 \ -2 \ 0 \ 2]$
and in this case $\Xi(A_1, A_2) = A_1^{-1} A_2^{-1}$
- (29) $1[12]2, 4[12]1 \& 2[12]1, 3[12]2 \& 5[12]1 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 2 \ 0 \ 2 \ 2]$
and in this case $\Xi(A_1, A_2) = -A_1 A_2^{-1}$
- (30) $1[12]2, 4[12]1 \& 2[12]4, 3[12]3 \& 5[12]4 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ -2 \ 0 \ 2 \ -2]$
and in this case $\Xi(A_1, A_2) = A_2^{-1} A_1$
- (31) $1[12]3, 4[12]4 \& 2[12]1, 3[12]2 \& 5[12]4 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 2 \ 0 \ -2 \ -2]$
and in this case $\Xi(A_1, A_2) = A_2 A_1^{-1}$
- (32) $1[12]3, 4[12]4 \& 2[12]4, 3[12]3 \& 5[12]1 \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ -2 \ 0 \ -2 \ 2]$
and in this case $\Xi(A_1, A_2) = -A_1^{-1} A_2$

Remark: Conditions $5[12]2, 5[12]3, 5[12]5, 2[12]\langle +1 \rangle, 2[12]\langle -1 \rangle, 3[12]\langle -1 \rangle, 4[12]\langle +1 \rangle, 4[12]\langle -1 \rangle$ are inconsistent; $3[12]\langle +1 \rangle$ is trivial.

We can say that conditions (1–12) are the primary conditions, (13–24) are composite conditions, (25–32) are extremal conditions.

3.10. Statement. For orthogonal invariant pseudoscalar FQ operations Ξ , the consistent constellations of principal hyperscaling properties are as follows:

- (1) $1[12]5 \Leftrightarrow 3[12]5 \rightsquigarrow I_{\text{ne}} = \{2, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ \hat{q}_2 \ 0 \ 0 \ 0]$
- (2) $2[12]1 \Leftrightarrow 3[12]2 \rightsquigarrow I_{\text{ne}} = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 2 \ 0 \ 0 \ 0]$
- (3) $2[12]4 \Leftrightarrow 3[12]3 \rightsquigarrow I_{\text{ne}} = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ -2 \ 0 \ 0 \ 0]$
- (4) $2[12]2 \Leftrightarrow 3[12]1 \rightsquigarrow I_{\text{ne}} = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 0 \ 2 \ 0 \ 0]$
- (5) $2[12]3 \Leftrightarrow 3[12]4 \rightsquigarrow I_{\text{ne}} = \{1, 4, 5\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 0 \ -2 \ 0 \ 0]$
- (6) $2[12]5 \Leftrightarrow 4[12]5 \rightsquigarrow I_e = \{3\}, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 0 \ \hat{q}_3 \ 0 \ 0]$
- (7) $1[12]5, 3[12]5 \& 2[12]1, 3[12]2 \Leftrightarrow 6[12]\langle +1 \rangle \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 2 \ 0 \ 0 \ 0]$
and in this case $\Xi(A_1, A_2) = \mathcal{A}_L(A_1, A_2)$
- (8) $1[12]5, 3[12]5 \& 2[12]4, 3[12]3 \Leftrightarrow 6[12]\langle -1 \rangle \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ -2 \ 0 \ 0 \ 0]$
and in this case $\Xi(A_1, A_2) = \mathcal{A}_R(A_1, A_2)$
- (9) $2[12]2, 3[12]1 \& 2[12]5, 4[12]5 \Leftrightarrow 7[12]\langle +1 \rangle \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 0 \ 2 \ 0 \ 0]$
and in this case $\Xi(A_1, A_2) = \mathcal{D}(A_1, A_2)$
- (10) $2[12]3, 3[12]4 \& 2[12]5, 4[12]5 \Leftrightarrow 7[12]\langle -1 \rangle \rightsquigarrow I_e = \emptyset, \hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 0 \ -2 \ 0 \ 0]$
and in this case $\Xi(A_1, A_2) = -\mathcal{D}(A_1, A_2)^{-1}$

We can say that conditions (1–6) are the primary conditions, (7–10) are extremal conditions.

3.11. Statement. For orthogonal invariant pseudoscalar FQ operations,

- an example for 3.10(1) is $\Xi = \mathcal{A}_C$ with $\hat{q}_2 = 0$, $\hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 0 \ 0 \ 0 \ 0]$;
- examples for 3.10(2) are $\Xi = \mathcal{D}\mathcal{L}^{-1}, \text{pol } \mathcal{D}\mathcal{L}^{-1}, -\mathcal{L}\mathcal{D}^{-1}, \mathcal{D}^x(\mathcal{L}^x)^{-1} = -\mathcal{L}^{\text{cx}}(\mathcal{D}^{\text{cx}})^{-1}, \mathcal{D}^{\text{cx}}(\mathcal{L}^{\text{cx}})^{-1} = -\mathcal{L}^x(\mathcal{D}^x)^{-1}$;
- examples for 3.10(3) are $\Xi = \mathcal{L}^{-1}\mathcal{D}, \text{pol } \mathcal{L}^{-1}\mathcal{D}, -\mathcal{D}^{-1}\mathcal{L}, (\mathcal{L}^x)^{-1}\mathcal{D}^x = -(\mathcal{D}^{\text{cx}})^{-1}\mathcal{L}^{\text{cx}}, (\mathcal{L}^{\text{cx}})^{-1}\mathcal{D}^{\text{cx}} = -(\mathcal{D}^x)^{-1}\mathcal{L}^x$;
- examples for 3.10(4) are $\Xi = \mathcal{D}^{\text{fx}}, \mathcal{D}^x, \mathcal{D}^{\text{cx}}$;
- examples for 3.10(5) are $\Xi = -(\mathcal{D}^{\text{fx}})^{-1}, -(\mathcal{D}^x)^{-1}, -(\mathcal{D}^{\text{cx}})^{-1}$;
- an example for 3.10(6) is $\Xi = \mathcal{A}_D = \text{pol } \mathcal{D}$ with $\hat{q}_3 = 0$, $\hat{\mathbf{P}}_{1\dots 5}^{[12]} = [0 \ 0 \ 0 \ 0 \ 0]$.

What we see is that only very few FQ operations are characterized by Clifford conservativity, principal hyperscalings and orthogonal invariance alone. The situation improves if one allows to combine them with scalar scaling conditions. For example, in case of 3.4(6), after only the index 3 left, a simple scalar homogeneity property (i. e. scalar scaling in \hat{r}_3) with $\hat{p}_3 = \alpha$ is sufficient to fix the FQ operation.

The statements above, in this form, are, of course, conjectural, and their proofs should be somewhat longish due, if not else, to the length of the statements themselves. However, certain restrictive aspects of them (like some inconsistencies) can be checked rather easily. The general picture they suggest is that hyperscaling conditions are heterogeneous, but they do not describe operations with any very specific properties but the arithmetically very simplest ones. In particular, hyperscaling conditions already limit first order behaviour severely (especially in the presence of orthogonal and transposition invariance properties), which makes them unsuitable for certain classes of operations.

3.12. Alternatively, one can try the combined hyperscaling conditions in \tilde{r}_4 and \tilde{r}_5 . In this case $L = 0$ can be assumed anyway. $\tilde{4}[s](J, 0, \alpha, \beta)$ and $\tilde{5}[s](J, 0, \alpha, \beta)$ reads as (with $\pm = +$

and $\pm = -$, respectively)

$$\begin{aligned}
\frac{1}{2}\hat{p}_4^{[s]} \pm \frac{1}{2}\hat{p}_5^{[s]} &= (\alpha + \beta)\hat{p}^{[s]} \\
\frac{1}{2}\hat{p}_{4,j,\dots}^{[s]} \pm \frac{1}{2}\hat{p}_{5,j,\dots}^{[s]} &= \alpha\hat{p}_{j,\dots}^{[s]} - \frac{1}{4}(J \pm 1)\hat{p}_{5*j,\dots}^{[s]} \mp \frac{1}{4}(J \pm 1)\hat{p}_{4*j,\dots}^{[s]} \\
\frac{1}{2}\hat{p}_{\dots,i,4}^{[s]} \pm \frac{1}{2}\hat{p}_{\dots,i,5}^{[s]} &= \frac{1}{4}(J \mp 1)\hat{p}_{\dots,i*5}^{[s]} \pm \frac{1}{4}(J \mp 1)\hat{p}_{\dots,i*4}^{[s]} + \beta\hat{p}_{\dots,i}^{[s]} \\
\frac{1}{2}\hat{p}_{\dots,i,4,j,\dots}^{[s]} \pm \frac{1}{2}\hat{p}_{\dots,i,5,j,\dots}^{[s]} &= \frac{1}{4}(J \mp 1)\hat{p}_{\dots,i*5,j,\dots}^{[s]} \pm \frac{1}{4}(J \mp 1)\hat{p}_{\dots,i*4,j,\dots}^{[s]} \\
&\quad - \frac{1}{4}(J \pm 1)\hat{p}_{\dots,i,5*j,\dots}^{[s]} \mp \frac{1}{4}(J \pm 1)\hat{p}_{\dots,i,4*j,\dots}^{[s]}.
\end{aligned}$$

Similarly, as before, they lead to various principal hyperscaling conditions, but they offer little more in the orthogonal invariant case:

3.13. Statement. *If an scalar or symmetric vectorial or pseudoscalar orthogonal invariant FQ operation Ξ satisfies a hyperscaling condition in \tilde{r}_4 or \tilde{r}_5 , then it is one of the following:*

- (1) $\Xi = 1$ (satisfies, for example, $\tilde{4}, \tilde{5}[0]\mathbf{2}, \mathbf{3}, \mathbf{5}$)
- (2) $\Xi = \text{Id}$ (satisfies, for example, $\tilde{4}[1]\mathbf{6}, \mathbf{8}$)
- (3) $\Xi = \mathcal{T}_{RR}$ (satisfies $\tilde{4}[1]\mathbf{7}$)
- (4) $\Xi = \mathcal{T}_{LL}$ (satisfies $\tilde{4}[1]\mathbf{9}$)
- (5) $\Xi = \mathcal{O}^{\text{afSy}}$ (satisfies, for example, $\tilde{4}[1]\mathbf{5}$)
- (6) $\Xi = \hat{q}_5 \text{Id} + (1 - \hat{q}_5)\mathcal{O}^{\text{fSy}}$ (satisfies, for example, $\tilde{5}[1]\mathbf{5}$)
- (7) $\Xi = A_R$ (satisfies, for example, $\tilde{4}[12]\mathbf{5}$)
- (8) $\Xi = A_L$ (satisfies, for example, $\tilde{5}[12]\mathbf{5}$)

One can also try hyperscaling condition with $L \neq 0$. This leads to some principal types with $L = \pm 1$ and a more complicated situation, but not much new in regard of orthogonal invariant FQ operations.

3.14. Or, we can carry out the same computations in the monoaxial regime. Then we deal only with variables $\hat{r}_3, \hat{r}_4, \hat{r}_5$ but the same principal types can be used. In fact, what happens is that we get the hyperscaling conditions in a much cleaner form as the principal hyperscaling conditions do not “glue” together as before. Nevertheless, “interactions” between them are possible if more of them are imposed. But even after the axial extensions we do not really arrive to essentially new examples compared to what we have seen.

So, while scalar scalings are much weaker, they can be used more flexibly than hyperscalings. Also, processes like axial extensions produce similar reductions but less restrictive.

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E-mail address: lakos@cs.elte.hu

DEPARTMENT OF GEOMETRY, EÖTVÖS LORÁND UNIVERSITY, PÁZMÁNY PÉTER S. 1/C, BUDAPEST, H-1117, HUNGARY